Math 141
Joseph C Foster
Summer 2017
Midterm 3

Name: Solutions

June 15th, 2017
Time Limit: 55 minutes

This exam contains 6 pages (including this cover page) and 12 questions. The total number of points is 100. You have 55 minutes to complete the exam.

Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.

No calculator/phone/smartwatch allowed under any circumstances. Place these items in your bag, out of reach. Cheating of any kind will not be tolerated and will result in a grade of zero.

Question	Marks	Score	Question	Marks	Score
1	5		8	12	
2	5		9	8	
3	5		10	12	
4	5		11	20	
5	5		12	12	
6	5				
7	6		Total	100	

1. (5 marks) True or False: If a function is integrable, then it is continuous.

   A. True

   B. False

2. (5 marks) Fill in the blank: The area under a curve over the interval $[a, b]$ can be calculated by first approximating it using n rectangles and then by calculating $\lim_{n\to\infty} \sum_{j=1}^{n}$ $F(a+j\frac{b-a}{n})\frac{b-a}{n}$.

3. (5 marks) Fill in the blank: If f is continuous on $[a, b]$ and F is *any* antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,dx =$ $F(b)-F(a)$.

For questions 4-6, choose the best answer. There is only one correct answer but you may choose up to *two*. If you choose two and one of the answers is correct, you will receive 2 marks.

4. (5 marks) Find: $\int \frac{1}{1+x^2}\,dx$.

   A. $\tan^{-1}(x) + C$

   B. $\sin^{-1}(x) + C$

   C. $\ln(1+x^2) + C$

   D. Does not exist.

5. (5 marks) Find: $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

   A. $\tan^{-1}(x) + C$

   B. $\sin^{-1}(x) + C$

   C. $\ln(\sqrt{1-x^2}) + C$

   D. Does not exist.

6. (5 marks) Calculate $F(x) = \int 2x + 1\,dx$ if $F(1) = 5$.

   A. $x^2 + x + 1$

   B. $x^2 + x + 2$

   C. $x^2 + x + 3$

   D. $x^2 + x + 4$

7. (6 marks) Evaluate $\int 24x^{11} + 24x^7 + 24x^5 + 24x^3 + 24x^2 + 24x + 24\,dx$.

$$= \frac{24}{12}x^{12} + \frac{24}{8}x^8 + \frac{24}{6}x^6 + \frac{24}{4}x^4 + \frac{24}{3}x^3 + \frac{24}{2}x^2 + 24x + C$$

$$= 2x^{12} + 3x^8 + 4x^6 + 6x^4 + 8x^3 + 12x^2 + 24x + C$$

8. (12 marks) Evaluate $\int \frac{1}{\sqrt{2x-\pi}} + 2\sec^2(2x-\pi)\,dx$

$u = 2x - \pi$
$du = 2dx$
$\frac{1}{2}du = dx$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}} + 2\sec^2(u)\,du$$

$$= \frac{1}{2}\int u^{-1/2} + 2\sec^2(u)\,du$$

$$= \frac{1}{2}\left[\frac{u^{-1/2+1}}{-1/2+1} + 2\tan(u)\right] + C$$

$$= \frac{1}{2}\left[2u^{1/2} + 2\tan(u)\right] + C$$

$$= u^{1/2} + \tan(u) + C$$

$$= (2x-\pi)^{1/2} + \tan(2x-\pi) + C$$

$$= \sqrt{2x-\pi} + \tan(2x-\pi) + C$$

9. (8 marks) Evaluate $\int_1^2 \frac{2x}{x^2+2}\,dx$

$$u = x^2+2 \qquad du = 2x\,dx \implies \int \frac{1}{u}\,du = \ln|u|\Big|_{x=1}^{x=2}$$

$$= \ln|x^2+2|\Big|_1^2 = \ln(x^2+2)\Big|_1^2$$

$$= \ln((2)^2+2) - \ln((1)^2+2)$$

$$= \ln(6) - \ln(3) = \ln(6/3) = \ln(2)$$

10. (12 marks) Evaluate $\int_0^{\pi/2} 4\pi\,(2\sin(x)\cos(x))^2\,dx$

Formula Sheet:

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$\sin^2(2x) = \frac{1}{2}(1-\cos(4x))$

$$= 4\pi\int_0^{\pi/2} (\sin(2x))^2\,dx$$

$$= 4\pi\int_0^{\pi/2} \sin^2(2x)\,dx$$

$$= 4\pi\int_0^{\pi/2} \frac{1}{2}(1-\cos(4x))\,dx$$

$$= 2\pi\int_0^{\pi/2} 1-\cos(4x)\,dx = 2\pi\left[x - \frac{1}{4}\sin(4x)\right]_0^{\pi/2}$$

$$= 2\pi\left[\pi/2 - 1/4\sin(2\pi) - (0 - 1/4\sin(0))\right]$$

$$= 2\pi\left[\pi/2 - 0 - (0-0)\right]$$

$$= 2\pi \cdot \pi/2$$

$$= \pi^2$$

11. Let $f(x) = 6x^2 + 17$.

(a) (16 marks) Find a formula for the Riemann sum obtained by dividing the interval $[2, 4]$ into n equal subintervals.

$$\frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$$

$$\sum_{j=1}^{n} f\left(a+j\frac{b-a}{n}\right)\frac{b-a}{n} = \sum_{j=1}^{n} f(2+j\tfrac{2}{n})\frac{2}{n} = \frac{2}{n}\sum_{j=1}^{n} 6(2+j\tfrac{2}{n})^2+17$$

$$= \frac{2}{n}\sum_{j=1}^{n} 6(4+\tfrac{8}{n}j+\tfrac{4}{n^2}j^2)+17 = \frac{2}{n}\sum_{j=1}^{n} 24+\frac{48}{n}j+\frac{24}{n^2}j^2+17$$

$$= \frac{2}{n}\sum_{j=1}^{n} 41+\frac{48}{n}j+\frac{24}{n^2}j^2 = \frac{2}{n}\sum_{j=1}^{n} 41 + \frac{2}{n}\sum_{j=1}^{n}\frac{48}{n}j + \frac{2}{n}\sum_{j=1}^{n}\frac{24}{n^2}j^2$$

$$= \frac{82}{n}\sum_{j=1}^{n} 1 + \frac{96}{n^2}\sum_{j=1}^{n} j + \frac{48}{n^3}\sum_{j=1}^{n} j^2$$

$$= \frac{82}{n}\cdot n + \frac{96}{n^2}\cdot\frac{n(n+1)}{2} + \frac{48}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 82 + 48(1+\tfrac{1}{n}) + 8(2+\tfrac{3}{n}+\tfrac{1}{n^2})$$

(b) (4 marks) Calculate the exact area underneath $f(x)$ over $[2, 4]$ using your formula calculated above.

$$\lim_{n\to\infty} 82+48(1+\tfrac{1}{n})+8(2+\tfrac{3}{n}+\tfrac{1}{n^2}) = 82+48+16$$

$$= 146.$$

12. Let $f(x) = \sqrt{x^2 - 1}$

(a) (5 marks) Calculate the volume of the solid formed by revolving the curve $y = f(x)$ about the x-axis from $x = 1$ to $x = 3$.

$$V = \int_a^b \pi y^2 dx = \int_1^3 \pi \sqrt{x^2-1}^2 dx = \pi \int_1^3 x^2 - 1 dx$$

$$= \pi \left[\frac{1}{3}x^3 - x\right]\Big|_1^3 = \pi\left[\frac{1}{3}(3)^3 - (3) - \left(\frac{1}{3}(1)^3 - (1)\right)\right]$$

$$= \pi\left[\frac{27}{3} - 3 - 1/3 + 1\right] = \pi\left[\frac{26}{3} - 2\right] = \pi\left[\frac{26}{3} - \frac{6}{3}\right]$$

$$= \frac{20\pi}{3}$$

(b) (7 marks) Calculate the volume of the solid formed by revolving the curve $y = f(x)$ about the y-axis from $y = 2$ to $y = 5$.

$$y = \sqrt{x^2 - 1} \implies y^2 = x^2 - 1 \implies x^2 = y^2 + 1$$

$$V = \int_a^b \pi x^2 dy = \int_2^5 \pi (y^2 + 1) dy = \pi \int_2^5 y^2 + 1 dy$$

$$= \pi\left[\frac{1}{3}y^3 + y\right]\Big|_2^5 = \pi\left[\frac{1}{3}(5)^3 + (5) - \left(\frac{1}{3}(2)^3 + (2)\right)\right]$$

$$= \pi\left[\frac{125}{3} + 5 - \frac{8}{3} - 2\right] = \pi\left[\frac{117}{3} + 3\right] = \pi\left[39 + 3\right]$$

$$= 42\pi$$